REFERENCES

1. KUKSIN S.B., Use of monotonic semigroups in the theory of perfect elastoplasticity. Uspekhi matem. nauk, 37,5, 1982.
2. KUKSIN S.B., Mathematical correctness of the boundary value problems of ideal elastoplasticity. Uspekhi matem. nauk, 38, 2, 1983.
3. FREUDENTHAL A. and GEIRINGER H., Mathematical Theories of Inelastic Continua. Moscow, Fizmatgiz, 1962.
4. KLYUSHNIKOV V.D., Mathematical Theory of Plasticity. Moscow, Izd-vo MGU, 1979.
5. MOSOLOV P.P. and MYASNIKOV V.P., Mechanics of Rigid-Plastic Media. Moscow, Nauka, 1981.
6. DUVAUT G. and LIONS J.L., Inequalities in Mechanics and Physics. Berlin, Springer-Verlag, 1976.
7. ALEKSEYEV V.M., TIKHOMIROV V.M. and FOMIN S.V., Optimal Control. Moscow, Nauka, 1979.
8. BREZIS H., Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert. Amsterdam L. North-Holland, 1973.
9. ROCHAFELLAR R.T., Convex Analysis. Princeton, N.Y., Princeton Univ. Press, 1970.
10. OVSYANNIKOV L.V., Lectures on the Fundamentals of Gas Dynamics. Moscow, Nauka, 1981.
11. TEMAM R., Navier-Stokes Equations. Amsterdam-Oxford, North-Holland, 1977.

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THE BOUNDARY LAYER IN THE FLOW OF A PLASTIC MEDIUM NEAR A ROUGH SURFACE*

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High-speed flow of an incompressible plastic medium past a rigid rough surface with slippage along it is investigated. It is assumed that the ratio of the yield point of the medium to the dynamic pressure in the flow is small. An asymptotic representation of the solution is constructed, based on the assumption due to Lavrent'ev that in the case of flows with such properties the principal parts of the velocity and stress fields are represented by the corresponding fields of an ideal fluid. Equations are obtained describing the flow in the boundary layer. Group-theoretic analysis is used to find their solution for flows past wedges and cones. The thickness of the boundary layer is estimated.

1. Let us consider the high-speed flow of an incompressible plastic medium past a fixed impermeable surface, with the particles slipping along the surface. The stresses in the medium satisfy the Mises plasticity condition with constant k /1/. We assume that

$$f = \sqrt{k/(\rho c^2)} \ll 1 \tag{1.1}$$

where ρ is the density of the medium and c is the characteristic velocity of the flow. Condition (1.1) means that the level of the stress deviator is small compared with the dynamic pressure of the flow. The condition can be satisfied in the flows possessing high deformation rates. It can be expected, by virtue of (1.1), that the velocity of stress fields will differ little from the corresponding fields in the analogous problem for a perfect fluid.

The perfect fluid model was widely used in /2/ in calculating the rigid, intensely deformed materials. In some cases, however, it is useful to know the magnitude of the correction related to the density of the medium. The problem was studied earlier in /3/ for several specific cases. In /4/ expansions of the velocity and stress fields over short distances from the boundary were constructed for the slow flows ($f = \infty$). The boundary layer in a viscoplastic medium was studied in /5—7/ assuming that no slippage of the particles along the boundary took place.

Below, using the results of /3/, we obtain equations describing the flow in the boundary layer, differing appreciably from the corresponding equations for viscous flow and /5—7/ and use them as the starting concepts. Group-theoretic analysis methods /8/ are used to obtain

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accurate, boundary layer-type solutions * (*Flitman L.M., On the boundary layer in certain problems of the dynamics of a plastic medium. Preprint In-ta problem mekhaniki Akad. Nauk SSSR, Moscow, No.150, 1980).

Let us introduce an Eulerian, curvilinear (x^1, x^2, x^3)-coordinate system attached to the surface S past which the flow takes place, the latter representing the coordinate plane $x^3 = 0$. Let (x^1, x^2) form a mesh on S, let the lines x^3 be orthogonal to it, and let the coordinate x^3 be equal to the arc length between the observer point and S. We take, as the scale of velocity, $\mathbf{u} = (u^1, u^2, u^3)$, pressure p, stress deviator $\mathbf{T} = \|\tau^{ij}\|$ and time t, the characteristic flow velocity c, the dynamic pressure ρc^2, and the plasticity constant k; l/c (l is the unit of length). Then the equation of conservation of mass and momentum will be

$$\operatorname{div} \mathbf{u} = 0, \; d\mathbf{u}/dt = -\operatorname{grad} p + f^2 \operatorname{div} \mathbf{T} \tag{1.2}$$

We shall assume that the medium is in the plastic state near the surface S, and is described by the Mises-Levi Eqs./1/

$$\mathbf{T} = N\boldsymbol{\varepsilon}, \quad 2I(\mathbf{T}) \equiv \tau_j^{\,i}\tau_i^{\,j} = 2 \tag{1.3}$$

Here $I(\mathbf{T})$ is the second invariant of the stress tensor, $N = N(\boldsymbol{\varepsilon})$ is the coefficient of proportionality of the deviators, obtained from the second relation of (1.3), $\boldsymbol{\varepsilon} = (\varepsilon^{ij})$ is the strain rate tensor which is a deviator by virtue of the first equation of (1.2),

$$2\varepsilon^{ij} = \nabla^i u^j + \nabla^j u^i$$

∇^i are the contravariant differentiation operators inthe space /9/.

Let us write for (1.2), (1.3) on S $(x^3 = 0)$ the condition of impermeability and define the tangential stress vector

$$u^3 = 0 \tag{1.4}$$

$$\tau^{3\alpha} = f^\alpha(x^1, x^2, t) \;\; (\alpha = 1, 2) \tag{1.5}$$

We can replace (1.5) by the condition /10/ (ν is the coefficient of friction)

$$\boldsymbol{\tau}^3 = \min(1, \nu(pf^{-2} - \tau^{33}))\,\mathbf{u}\,|\mathbf{u}|^{-1}$$

This condition may hold when the normal stress on S is compressive. It means that the slippage along S takes place either in the dry friction mode (here $|\boldsymbol{\tau}^3| < 1$) or $|\boldsymbol{\tau}^3| = 1$ i.e. it attains its maximum value allowed by the condition of plasticity. When ν are not very small and conditions (1.1) hold, the condition represents a special case of (1.5), provided that the direction of the velocity $\mathbf{u}$ on S is known.

2. Assumption (1.1) leads to the appearance of a small parameter f^2 in (1.2). When $f = 0$ (1.2) becomes an equation for an ideal fluid, which agrees with what was said in Sect.1. We shall denote such a velocity field satisfying condition (1.4) by $\mathbf{v}$, and the corresponding pressure field by q. Then, using the known field $\mathbf{v}$ we can easily find $\mathbf{T}$ from (1.3). After this we can find the correction to the velocity from (1.2) by linearizing them about $\mathbf{v}$; the correction to the velocity and pressure will be of the order of f^2.

Let us write the total fields $\mathbf{u}$ and p in the form

$$\mathbf{u} = \mathbf{v} + f^2\mathbf{v}_1, \; p = q + f^2 q_1 \tag{2.1}$$

Then, in accordance with what was said above, we have the following relations from (1.2) for steady state flow in an irrotational field $\mathbf{v}$:

$$\operatorname{div} \mathbf{v} = 0, \; \operatorname{grad}(q_1 + \mathbf{v}\mathbf{v}_1) - [\mathbf{v} \times \operatorname{rot} \mathbf{v}_1] = \operatorname{div} \mathbf{T} \tag{2.2}$$

The Eqs.(2.2) for the corrections are linear, and the quantity $\operatorname{div} \mathbf{T}$ found on $\mathbf{v}$ plays the part of the mass forces. Here we have the situation studied in /3/ for the specific cases. It is clear that the stream lines of the field $\mathbf{v}$ are the characteristics of system (2.2). Therefore, we can specify for (2.2) on the surface S only condition (1.4), but not (1.5). The procedure shown can be used to find the corrections of order f^4 and higher. We can construct in this manner the outer asymptotic expansion for the solution /11/ which does not satisfy condition (1.5). To satisfy (1.5), we must construct the inner asymptotic expansion near S, i.e. the boundary layer (BL).

3. In deriving the equations we shall use the results of /3/ to formulate the basic hypotheses. We shall write the required fields $\mathbf{u}$ and p in the form

$$\mathbf{u} = \mathbf{v} + f\mathbf{w}, \; p = q + f^2 s \tag{3.1}$$

Here $\mathbf{v}$ and q are the fields in the ideal fluid corresponding to $f = 0$ and w, s are the corrections.

The representation (3.1) presupposes the slippage of the medium particles along the surface S, and means that the fields $\mathbf{v}$ and q provide the main contribution towards the quantities sought. We see from relations (3.1) the deviation from the classical formulation of the problem of a viscous fluid where the correction for $\mathbf{v}$ is of order $\mathbf{v}$. The difference is due to the fact that there is no slippage in the viscous model and no constraints are imposed on the magnitude of the tangential stress. In the case in question the slippage is allowed, since the tangential stresses are restricted by condition (1.3) and relation (1.1) holds by definition.

In accordance with what was said in Sect.2, we shall assume that $\mathbf{w}, s, \mathbf{T}$ vary rapidly near S, in the sense that their derivatives in x^3 are large. We shall write, as in /3/,

$$x^3 = fz,\ \mathbf{w} = \mathbf{w}(x^1, x^2, z, t),\ s = s(x^1, x^2, z, t),\ \mathbf{T} = \mathbf{T}(x^1, x^2, z, t) \tag{3.2}$$

Let us denote by e^{ij} the deformation rates calculated over the field $\mathbf{v}$ according to (1.3). Let us also assume that they are not all zero. This clearly does not imply that $\mathbf{v}$ provides the main contribution towards ε. The hypothesis just suggested means that we consider only those $\mathbf{v}$, such that particles moving near the surface S are deformed. In other words, we assume that energy is dissipated near S. Thus we eliminate from our discussion the problems in which a flow zone with constant velocities exists around S, e.g. the selfsimilar problem on skew shock /12/ which has no boundary layer. Henceforth, we shall assume that the fields $\mathbf{v}$ and q are known.

We also note that the following condition follows from (1.4) and (3.1):

$$w^3 = 0 \quad (z = 0) \tag{3.3}$$

and we have conditions (1.5) for $\mathbf{T}$.

4. Using the assumptions of Sect.3, remembering that $\mathbf{v}$ and q satisfy Eqs.(1.2) at $f = 0$ and neglecting terms of order higher than f^2, we can eliminate from (1.2), (1.3) some of the unknowns and obtain the following BL equations near S:

$$w_{,t}^{\alpha} + v^{\beta}\nabla_{\beta}w^{\alpha} + w^{\beta}\nabla_{\beta}v^{\alpha} - z\varphi w_{,z}^{\alpha} = \tau_{3,z}^{\alpha} \tag{4.1}$$

$$w_{,z}^{\alpha} + 2e_3^{\alpha} = 2\psi\tau_3^{\alpha}(1 - \tau_3^{\beta}\tau_{\beta}^{3})^{-1/2} \quad (\alpha, \beta = 1, 2) \tag{4.2}$$

The index following the comma denotes differentiation with respect to the corresponding variable, ∇_{β} is the covariant differentiation operator on S [9], e_3^{α} are the deformation rates calculated over the field $\mathbf{v}$ and taken on S. The functions φ and ψ also depend only on x^1, x^2 and are expressed in terms of v

$$\varphi = \nabla_{\beta}v^{\beta}, \quad 2\psi^2 = e_{\beta}^{\alpha}e_{\alpha}^{\beta} + e_{\alpha}^{\alpha}e_{\beta}^{\beta} \quad (\alpha, \beta = 1, 2) \tag{4.3}$$

All characteristics of the field $\mathbf{v}$ are taken with $x^3 = 0$.

Projecting (1.1) onto the normal to S we obtain, in addition to (4.1), (4.2),

$$(-s - \tau^{33})_{,z} = 0$$

The above relation means that the total normal stress σ^{33} is constant near the surface S along a fixed normal, the same as in the outer expansion described in Sect.2.

We take (1.5) as the boundary conditions for (4.1), (4.2) at $z = 0$, and at $z = \infty$ we have

$$w^{\alpha} = 0 \quad (\alpha = 1, 2) \tag{4.4}$$

Relation (4.4) represents the condition for matching the solution in the BL to the outer expansion. If the condition is satisfied, then the quantity $\mathbf{T}$ from the BL tends clearly, as $z \to \infty$ to the quantity $\mathbf{T}$ from the outer expansion defined over $\mathbf{v}$ in accordance with what was said in Sect.2. Next we consider a steady flow, using the stream lines of the field $\mathbf{v}$ as the coordinate lines x^1. Equations (4.2) remain unchanged, and (4.1) take the form

$$v^1\nabla_1 w^{\alpha} + w^{\beta}\nabla_{\beta}v^{\alpha} - z\varphi w_{,z}^{\alpha} = \tau_{3,z}^{\alpha} \tag{4.5}$$

We see from (4.2) and (4.5) that the surfaces $x^2 = \text{const}$ represent the characteristics of this system, and the variable x^2 appears only as a parameter. In other words, we can study the flow in the BL along every stream line of the field $\mathbf{v}$ independently of each other. It is this factor that distinguishes Eqs.(4.2) and (4.5) from the corresponding equations of hydrodynamics. The circumstance mentioned satisfies the problem and makes the results of the plane and axisymmetric solutions given in /3/ and below, more representative.

If the field $\mathbf{v}$ is irrotational, then

$$e_3^{\alpha} = v^1 b_1^{\alpha} \tag{4.6}$$

Here b_1^{α} is a component of the tensor consisting of the coefficients of the second quadratic form of S.

Let S be a cylinder or a surface of revolution, and the flows $\mathbf{v}$ and $\mathbf{w}$ be plane or axisymmetric, let x^1 denote the arc length along the direction of the cylinder or meridian measured from the stagnation point, and, in addition, let $\tau_3^2 = 0$. Then, writing $v^1 = v, w^1 = w, \tau_3^1 = \tau$ we obtain, from (4.5), (4.2), (4.6),

$$(vw)_{,1} - z\varphi w_{,z} = \tau_{,z}, \quad w_{,z} - 2v/R = 2\psi (1 - \tau^2)^{-1/2}\tau \tag{4.7}$$

Here R is the radius of curvature of the directrix or meridian. In accordance with (4.3) we have $\varphi = \psi = v_{,1}$ for the plane problem, and $\varphi = v_{,1} + B^{-1}B_{,1}v$, $\psi^2 = \varphi^2 - B^{-1}B_{,1}v_{,1}v$ for the axisymmetric problem. Here B is the coefficient of the first quadratic form of S taken in the form $(ds)^2 = (dx^1)^2 + B\,(dx^2)^2$.

Eqs. (4.7) are analogous to the equations studied in /3/.

Let us introduce the notation

$$x = \int_{x^1}^{x^1} \psi(\alpha) v(\alpha)\, d\alpha, \quad y = \psi(x^1) v(x^1) z, \quad u = vw \tag{4.8}$$

$$F(\tau) = 2\tau (1 - \tau^2)^{-1/2}, \quad \omega(x) = (v\psi)^{-2} (\varphi\psi - (v\psi)_{,1})$$

$$k(x) = 2v (\psi R)^{-1}$$

System (4.7) in the new variables takes the form

$$u_{,y} = F(\tau) + k(x), \quad u_{,x} = \tau_{,y} + y\omega(x) u_{,y} \tag{4.9}$$

and we shall study this system below.

In the case of a plane flow within a right angle whose side $x^1 = 0$ is smooth and $x^3 = 0$ is rough, we use as v the well-known expression for an ideal fluid /13/ to reduce system (4.9) to the following non-linear parabolic equation:

$$\tau_{,yy} = 2(1 - \tau^2)^{-1/2} \tau_{,x}$$

Symmetry considerations and conditions (1.5) and (4.4) enable us to formulate the boundary value problem for system (4.8) in the form

$$u = 0\ (x = 0), \quad \tau = \tau_0(x)\ \ (y = 0), \quad u = 0\ (y = \infty) \tag{4.10}$$

We also note that for the BL around a cone with a flow of a hardening plastic medium with a shock wave attached to the tip of the cone (the medium is incompressible behind the shock), (4.9) yields the equations which have been obtained and studied in /3/.

5. The BL equations (4.1), (4.2) and their special case (4.9) with the boundary conditions (4.10) were obtained assuming that such a layer exists. The author has no proof of this assumption. Below we give particular solutions of problem (4.9) and (4.10) which show that, at least in the cases studied, the assumption does not lead to a contradiction. The solutions themselves are interesting, and give, in addition, an idea of the possible thickness of the BL and the manner in which the solutions decrease. Some solutions of (4.9) can be found using group-theory methods of analyzing differential Eqs./8/. We find that if $\omega(x)$ and $k(x)$ satisfy the conditions

$$\xi\omega_{,x} + \beta\omega = 0, \quad (\xi k_{,x})_{,x} = \xi\omega k_{,x}, \quad \xi = \alpha + \beta x \tag{5.1}$$

(α and β are arbitrary constants), then Eqs. (4.9) admit of the group.

Having found the group invariants, we can find excact solutions of the non-linear system (4.9). Such solutions are given in /3/ for $\omega = \text{const}$ and $k = 0$. Let us consider the case

$$\omega = \nu/x, \quad k = 0 \tag{5.2}$$

Such ω and k occur in plane flows past wedges and in axisymmetric flows past cones. For the wedges with half-angle $\lambda\pi$ we have, in accordance with (4.8) and the well-known expressions for the velocities of an ideal fluid flowing past a wedge /13/,

$$\nu = \frac{1}{2\lambda} - 1, \quad x = \frac{1}{2}(x^1)^{1/(1/2-\nu)}, \quad y = \frac{z}{2\nu + 1}(2x)^{1/2-\nu} \tag{5.3}$$

Similarly, for a cone with half-angle $\lambda\pi$ we have

$$\nu = \frac{2-\mu}{2\mu}, \quad x = \frac{a}{2\mu}(x^1)^{2\mu}, \quad y = az(x^1)^{2\mu-1} \quad (a^2 = 1 + \mu + \mu^2) \tag{5.4}$$

Here the quantities λ and μ are connected by the relation (P_α^1 is the associated Legendre function of the first kind)

$$P_{1-\mu}^1(\cos\lambda\pi) = 0 \tag{5.5}$$

In the case of cones with an acute angle ($\lambda \ll 1$), the approximate solution (5.5) will be $2\mu = (\pi\lambda)^2$. When $\lambda = 1/2$ (axisymmetric flow impinging on a plane), (5.5) yields $\mu = 1$.

Using the fact that expressions (5.2) satisfy the conditions for the existence of the group (5.1), we shall seek, in accordance with /8/, the partly invariant solutions of system (4.9) in the form (b is an arbitrary constant)

$$\tau = \tau(t),\ u = x^{1/2}\theta(t),\ t = x^{-1/2}(y + bx^{-\nu}) \tag{5.6}$$

From (4.9) we obtain for τ and θ the following system in ordinary derivatives:

$$\tau_{,t} + (2\nu + 1)\, t\tau (1 - \tau^2)^{-1/2} = \theta/2,\ \theta_{,t} = 2\tau (1 - \tau^2)^{-1/2} \tag{5.7}$$

and, in accordance with (1.5), (4.4) and (5.10), we must set the following conditions for system (5.7):

$$\tau(t_0) = \tau_0 > 0,\ \theta(\infty) = 0 \tag{5.8}$$

We note that the selfsimilar problem of determining the field about an expanding and rotating cylinder studied in /3/, reduces to (5.7), (5.8).

N.K. Balabaev communicated to the author the proof of the fact that when $\nu > -1/2$, the problem (5.7), (5.8) has a unique solution. It was also shown that τ and θ tend monotonically to zero, decreasing and increasing respectively ($\tau > 0,\ \theta < 0$). He also solved the problem for $\nu = 0$, $1/2$ and different τ_0. He found that when t varies from zero to unity, τ decreases by at least one order of magnitude.

Knowing that a solution of the problem (5.7), (5.8), tending to zero at infinity, exists, we can obtain, as in /3/, its asymptotic expression for large t

$$\tau = \tau_1 t^{-1/(2\nu+1)} \exp\left(-\left(\nu + \frac{1}{2}\right)t^2\right),\quad \theta = -t^{-1}\left(\nu + \frac{1}{2}\right)^{-1}\tau(t) \tag{5.9}$$

We have analogous results for other ω and k satisfying conditions (5.1). What was said above, leads us to believe that the problem (4.9), (4.10) has solutions of the BL-type when $\omega > 0$.

6. The results of Sect.5 make possible the study of the BL appearing in flows past wedges and cones. Let us consider the plane and the axisymmetric case of a flow impinging on a plane (the half-angle of the wedge or cone is $\pi/2$). Formulas (5.3) and (5.4) yield here $\nu = 0$ and $\nu = 1/2$ respectively.

The change in τ in the BL can be conveniently assessed by studying the distribution of the level lines of this function. In the (x^1, z) plane we obtain from (5.3), (5.4), (5.6) the following expression for the level lines $t = \text{const}$:

$$z + b(x^1)^{-(2\nu-1)} = (8\nu + 2)^{-1/2}\, t \tag{6.1}$$

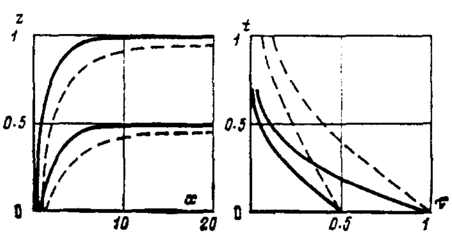
When $b > 0$, they are monotonically increasing functions $z(x^1)$ with the common vertical asymptote $x^1 = 0$ and horizontal asymptotes $z = t(8\nu + 2)^{-1/2}$. When $t = 0$, the x axis represents such an asymptote. This means that the function τ is close, at the boundary of the region in question, to a constant, and this can be dealt with by putting, in particular, $\tau_0 = 1$. It tends to zero near $x^1 = 0$ with all its derivatives

$$\tau = \tau_2 x^1 \exp\left(-b_1 (x^1)^{-2(2\nu+1)}\right)$$

(which follows from (5.6) and (6.1)). Relation (5.9) and the form of the contour lines $\tau(x_1, z)$ (6.1) imply that the conditions hold at infinity.

Thus the solution of (5.7), (5.8) satisfies the BL equations and its boundary values are close to the required conditions (4.10). We can also say that this solution "corrects" the boundary condition in the neighbourhood of the stagnation point $x^1 = z = 0$. Condition (1.5) is imposed where slippage occurs, but it is difficult to speak of slippage near the stagnation point. Judging from the zero approximation to v, a particle belonging to the medium present at some instant at the point $x^1 = z = 0$. moves away from it by a finite distance over an infinite period. Therefore the solution of (5.7), (5.8) obtained describes the phenomenon correctly.

The figure shows the relation $\tau(x^1, z)$ calculated by N.K. Balabaev. On the left we have the contour lines of τ (the solid lines correspond to $\nu = 1/2$ and the dashed lines to $\nu = 0$), and on the right we have the relations $\tau = \tau(t_1)$ where $t_1 = (8\nu + 2)^{-1/2} t$, obtained by numerical integration of the problem (5.7), (5.8) for two values of τ_0. The value of τ at some contour line is found as follows. The magnitude of the ordinate z of the horizontal asymptote of the contour line in question is used to obtain, according to (6.1), $t_1 = z$. After this the curve $\tau = \tau(t_1)$ is used to find the corresponding value of τ.



The existence of a horizontal asymptote to the contour line τ means that in the flow in question the BL has finite thickness, unlike a viscous fluid.

The figure shows that when t_1 varies from zero to 1.5, the quantity τ decreases more than tenfold. If we take, as the thickness of the BL, the distance at which such decrease occurs, then, taking into account (3.2), we can say that the thickness of the BL is approximately equal to f.

Similar results are obtained in the course of the study of BL on wedges and cones. They cease to be valid however at considerable distances from the tip. This is due to the fact that the deformations in the zeroth approximation decrease rapidly, and the condition that they must not be too small adopted in Sect.3, no longer holds.

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REFERENCES

1. LEIBENZON L.S., Course in the Theory of Elasticity. Moscow-Leningrad, Gostekhizdat, 1947.
2. LAVRENT'EV M.A. and SHABAT B.V., Problems of Hydrodynamics and their Mathematical Models. Moscow, Nauka, 1973.
3. FLITMAN L.M., On the boundary layer in certain problems of the dynamics of a plastic medium. Izv. Akad. Nauk SSSR, MTT, 1, 1982.
4. SOKOLOVSKII V.V., On the equations of plastic flow in a boundary layer. PMM 20, 3, 1956.
5. ASTRAKHAN I.M., On the equations of motion of a viscoplastic fluid in a boundary layer on an arbitrary surface. Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie, 2, 1960.
6. MOSOLOV P.P. and MYASNIKOV V.P., The boundary layer in the problem of the longitudinal motion of a cylinder in a viscoplastic medium. PMM 38, 4, 1974.
7. MOSOLOV P.P. and MYASNIKOV V.P., Mechanics of Rigid-Plastic media. Moscow, Nauka, 1981.
8. OVSYANNIKOV L.V., Group Analysis of Differential Equations. Moscow, Nauka, 1978.
9. VEKUA I.N., Fundamentals of Tensor Analysis and Theory of Covariants. Moscow, Nauka, 1978.
10. GRIGORYAN S.S., A new law of friction and the mechanism of large-scale mountain avalanches and landslides. Dokl. Akad. Nauk SSSR, 244, 4, 1979.
11. COLE J D., Perturbation Methods in Applied Mathematics. Blaisdell, Waltham, Mass., 1968.
12. SKOBEYEV A.M., On a plane, elastoplastic wave. PMM 29, 3, 1965.
13. KOCHIN N.E., KIBEL' I.A. and ROZE N.V., Theoretical Hydrodynamics. Pt. 1, Moscow, Fizmatgiz, 1963.

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